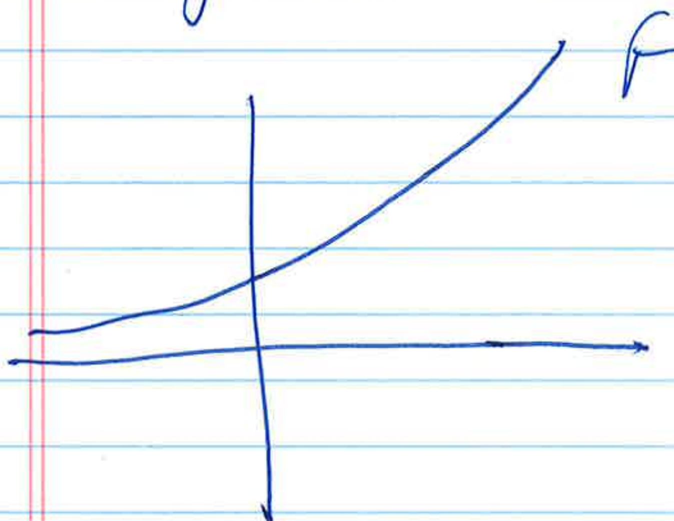
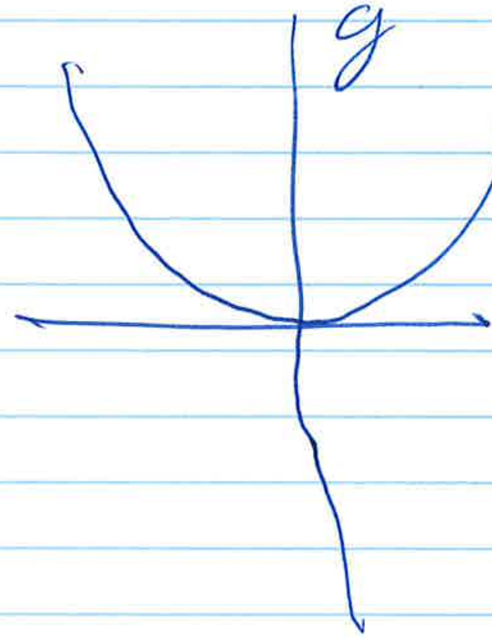
Math 151 - 2015XB - Week 4 Tues.

Today: Exponential Functions
Yesterday: Review of Algebra concerning exponents

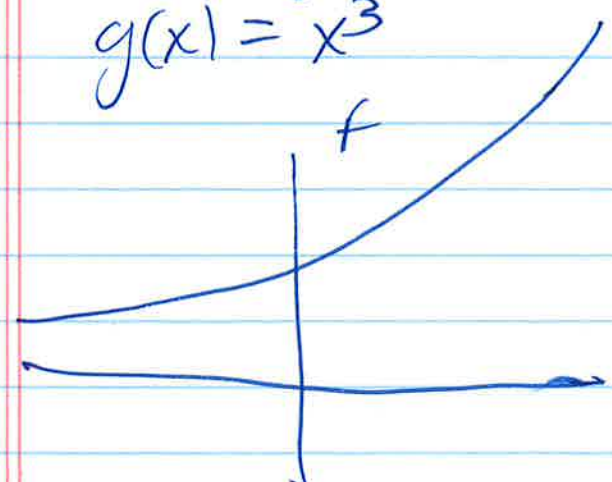
Two functions

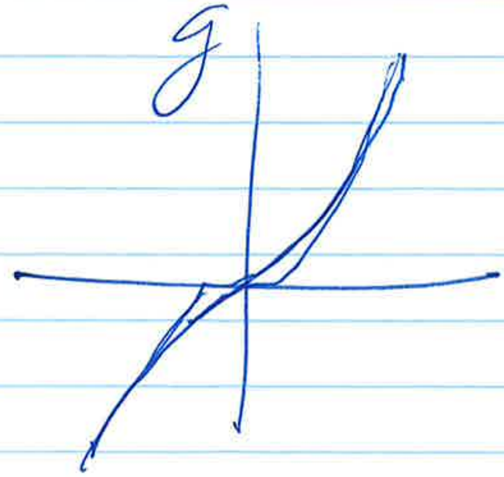
$f(x) = 2^x$
$g(x) = x^2$



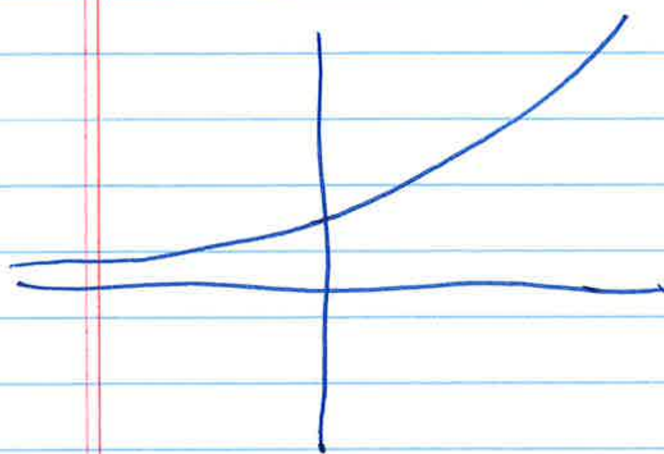


$f(x) = 3^x$
$g(x) = x^3$



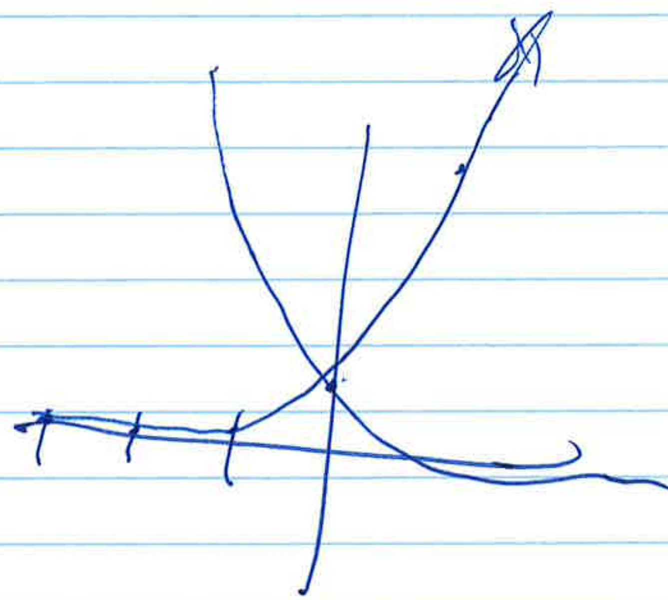


Exponential Function all look similar



They look like one of two thing or 3 really

$F(x) = 10^x$



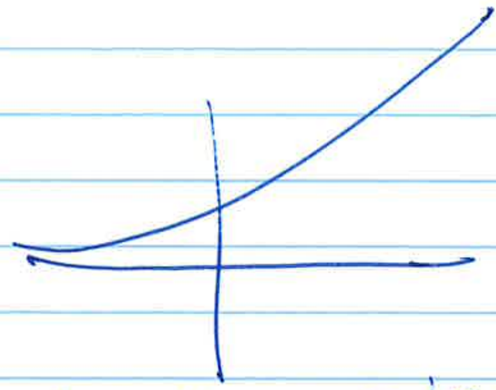
$F(0) = 1$
$F(1) = 10$
$F(2) = 100$
$F(3) = 1000$
$F(-1) = .1$
$F(-2) = .01$
$F(-3 = .001$

$F(x) = \left(\frac{1}{10}\right)^x = 10^{-x}$

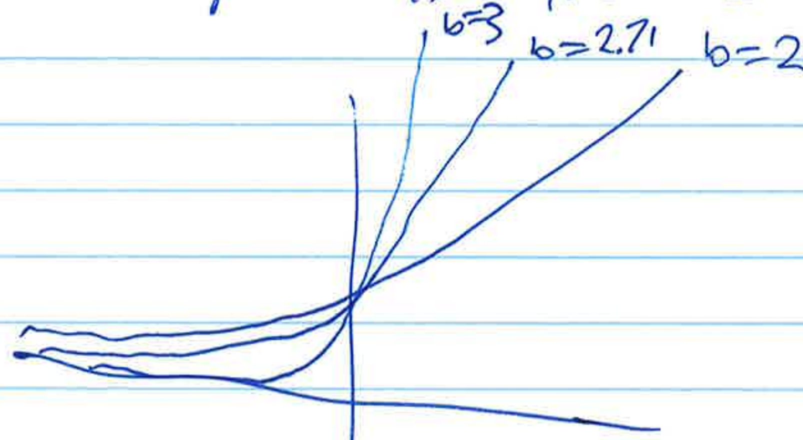
$F(0) = 1$
$F(1) = \frac{1}{10} = .1$
⋮

$F(x) = b^x$

IF $b > 1$

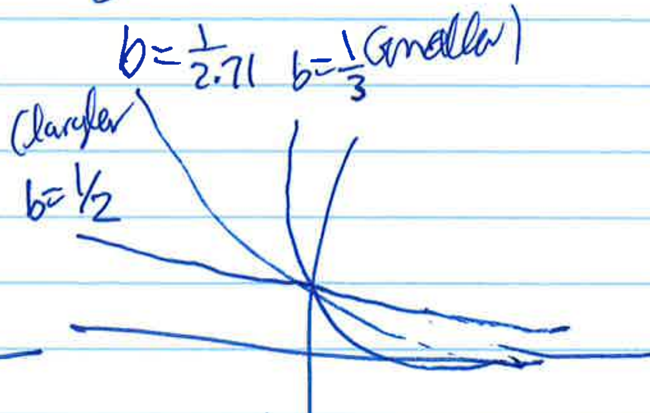


The larger b is the steeper



larger b decays faster on left
as well as grows faster on right

IF $b < 1$ then graph is reflected across y-axis



In fact b and 1/b are perfect reflections of each other

10 and 1/10

Turns out you can use any base b you want.

Let say your expenential function is

$y = \left(\frac{1}{10}\right)^x$ and you want to use base 10 instead of $\frac{1}{10}$.

Easy $y = 10^{-x}$

Lets say you wan to use 5

Solve $5^k = \frac{1}{10}$ for k

$$k \log 5 = \log \frac{1}{10}$$

$$k = \frac{\log 1/10}{\log 5}$$

$$y = \left(\frac{1}{10}\right)^x = \left(5^k\right)^x = 5^{kx}$$

$$= -1.43067$$

$$y = 5^{-1.43067x}$$

irrational number

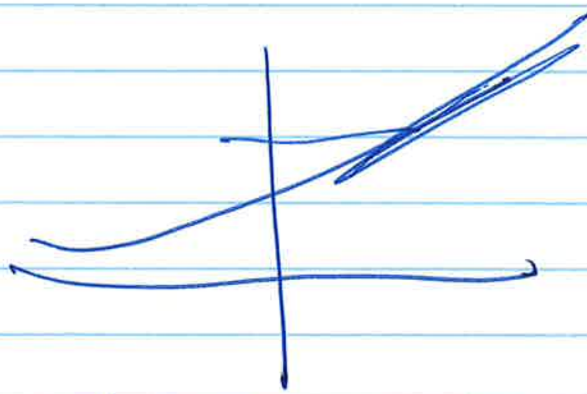
Which base to use?

It turns out one base in particular is preferred when you do calculus: e

Reason: Formulas are simplest in that base when you do calculus

Has to do with the fact that the slope of the function

$$f(x) = e^x \text{ is } e^x$$



e is the unique base with this property.

Another base?

The slope of $f(x) = e^{kx}$ is ke^{kx}

The complication in the formula arise here

e is irrational and that's unfortunate

But ~~you have~~ as I have shown you have to work with irrationals even with rational bases

So growth and decay application tend to work with $y = e^{kt}$ instead of $y = b^t$ (here the independent variable is t for time).

If we have $y(t) = e^{kt}$

Then $y(0) = e^{k \cdot 0} = 1$

this function always has value 1 at time 0.

We are going to make one more modification to make this more general: multiply by C

What if you don't have 1 dollar in you bank accont initially

You have \$1000, Then $C = 1000$

$y(t) = 1000 e^{kt}$

k is the relative growth rate

We say y is growing continuously at a relative growth rate k means $y = Ce^{kt}$ where C is the value of y at time 0

Do eg 2

Matched 2

(A) $55e^{(1.386)(.85)} = 178$ or 179

(B) $55e^{(1.386)(7.25)} = 1{,}271{,}659$

Eg 3

$A_0 = 500$

$A = 500e^{-0.000124 \cdot 15000}$

$= 77.83$

(B) $\cancel{\frac{250}{500} = \frac{500}{500} e^{-(0.000124)t}}$

$\cancel{\ln \frac{1}{2} = -0.000124t}$

How would you solve for half life

plug in $A = 250$ solve for t

not doing Eg 4

pg 8

Interest (compound)

principle

interest rate 0.07 for 7%

$$A = P\underbrace{(1+r)}^{t}$$

r is the annual interest rate

1+r is the growth factor

$b = 1+r$

usually under annual compounding t is an integer but it doesn't have to be

We can put this into base e

$$A = Pe^{kt}$$

$\underbrace{e^k}$ is growth factor

k is called ~~the~~ continuous compounding ~~growth~~ interest rate under

Book uses same letter for

annual and continuous

Other books do as well

This gets confusing!

More to be said: can compound every month every week every day, every second, Continuous compounding is the limit of this process. I'll come back to this in Chapter 3.

not doing eg 5

Matched 6:

Pg 9

$$5000 \cdot e^{.09 \cdot 5} = 7{,}841.56$$

Formula for compounding m times per year

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Priciple P

Annual rate r

Amount in account A at time t in years

t time in years

m is number of times compounded per year

Eg: m=12 compunded monthly

$$\underbrace{\left(1 + \frac{r}{m}\right)^{m}}_{\text{annual growth factor}}$$

gets closer and closer to e^{k} as m gets large.

Exponential functions and logarithmic functions: —closely related

Infact $\log_{10}(x)$ is the <u>inverse</u> of 10^x

↳ usually written $\log(x)$

In fact they are inverses of each other

$\log_e(x)$ is <u>inverse</u> of e^x

↳ usually written $\ln(x)$

In fact they are inverses of each other

What are inverses;

Think addition / subtraction
multiplication / division
exponential / logarithm

Each undoes ~~the~~ its complement

If you add a number and then subtract the same number you get back to the same thing.

In this way we are thinking of inverse operations

We can think of inverse functions

$f(x) = x + 5$ add 5 to x
$g(x) = x - 5$ subtract 5 from x

Find $f(g(x)) = g(x) + 5 = x - 5 + 5 = x$

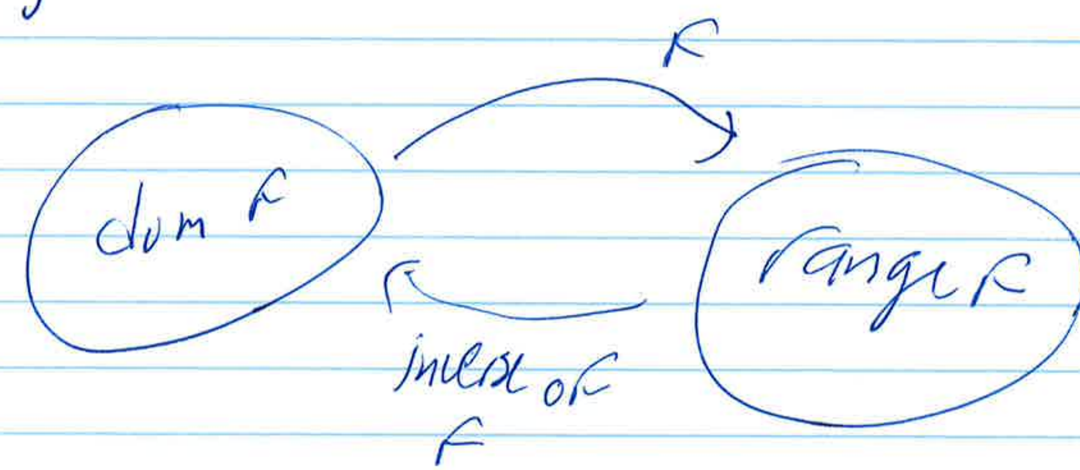
Find $g(f(x)) = f(x) - 5 = x + 5 - 5 = x$

Two functions are inverses of each other if

$$f(g(x)) = g(f(x)) = x$$

There's more to be said here

The inverse of f maps the range of f to the domain of f



~~Both of~~ So we are reversing y and x

$y = x + 5$ / $x = y - 5$ y becomes the indep variable

solve for x $\varphi(y) = y - 5$

This is how you find the inverse

$y = F(x)$

Solve for $X = g(y)$

$y = x + 5$ addition
$x = y - 5$ subtraction

$y = 2x$ multiplication
$x = \frac{1}{2} y$ division

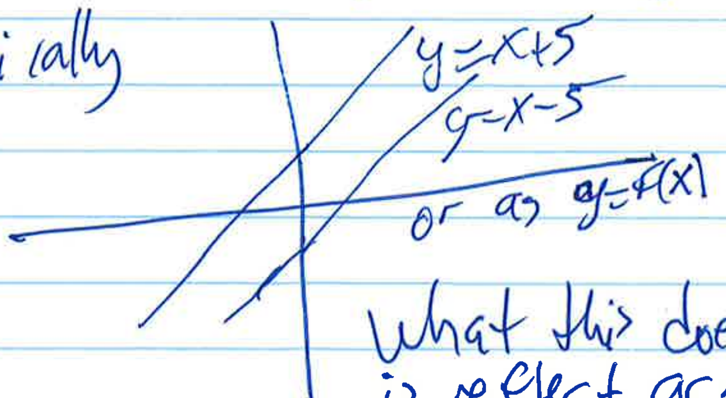
Inverse functions are sometimes written with y as the independent variable as shown or as

$x = \cancel{4}F(y) = \frac{1}{2} y$
$y = f(x) = \frac{1}{2} x$

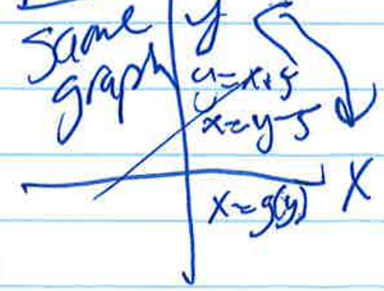
same function different names for vars

OR with X as dependent variable (just renaming variables)

Grapically

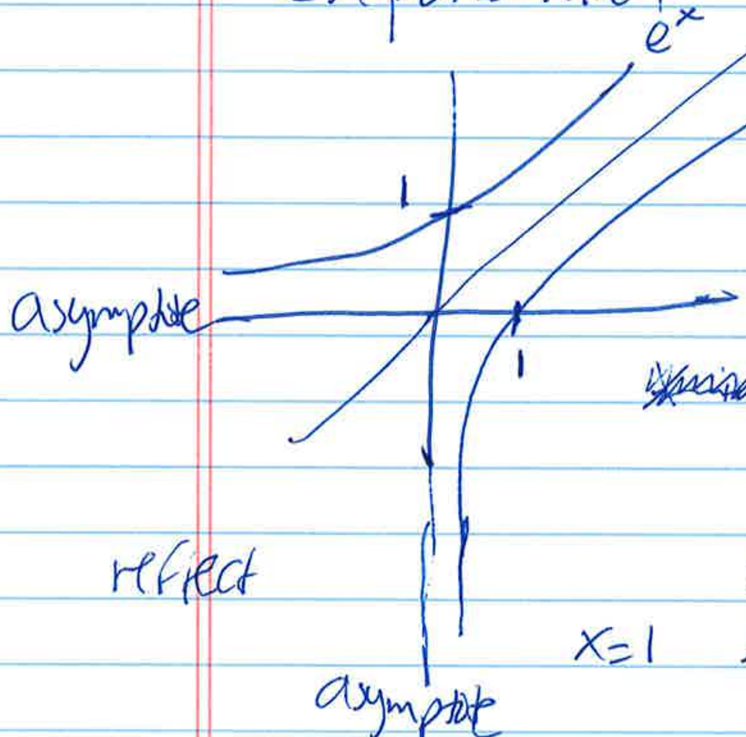


Flip axes



What this does is reflect across 45° line

Basically if this is the exponential



this is the logarithm

e^x y-intercept 1

$x=0$ / $e^x=1$

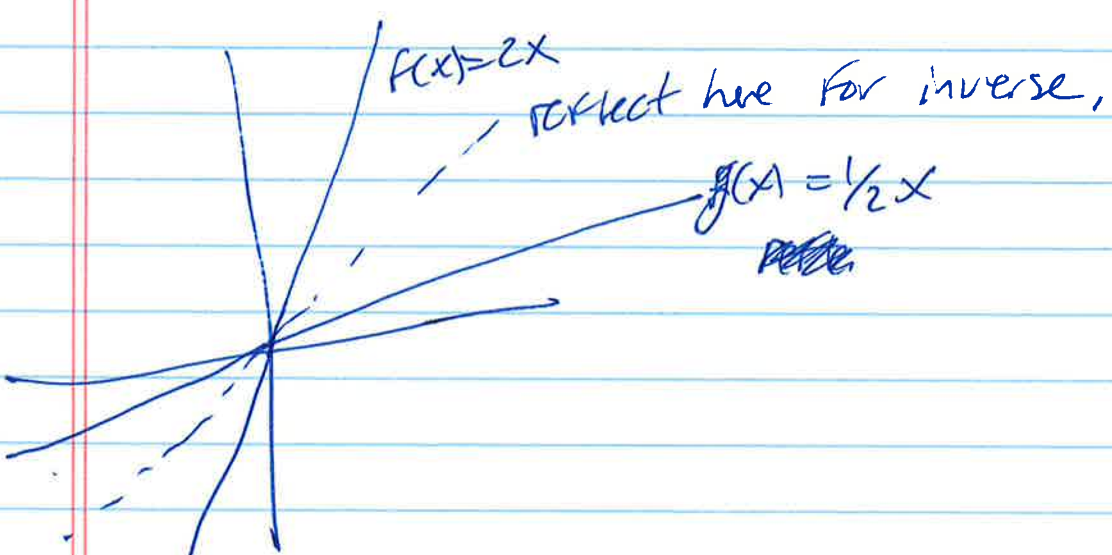
no x intercepts

$\ln(x)$ x intercept is 1

$x=1$ $\ln(x)=0$

$\ln(1)=0$

no y intercept.



A function must have "each element of domain maps to one element of range"

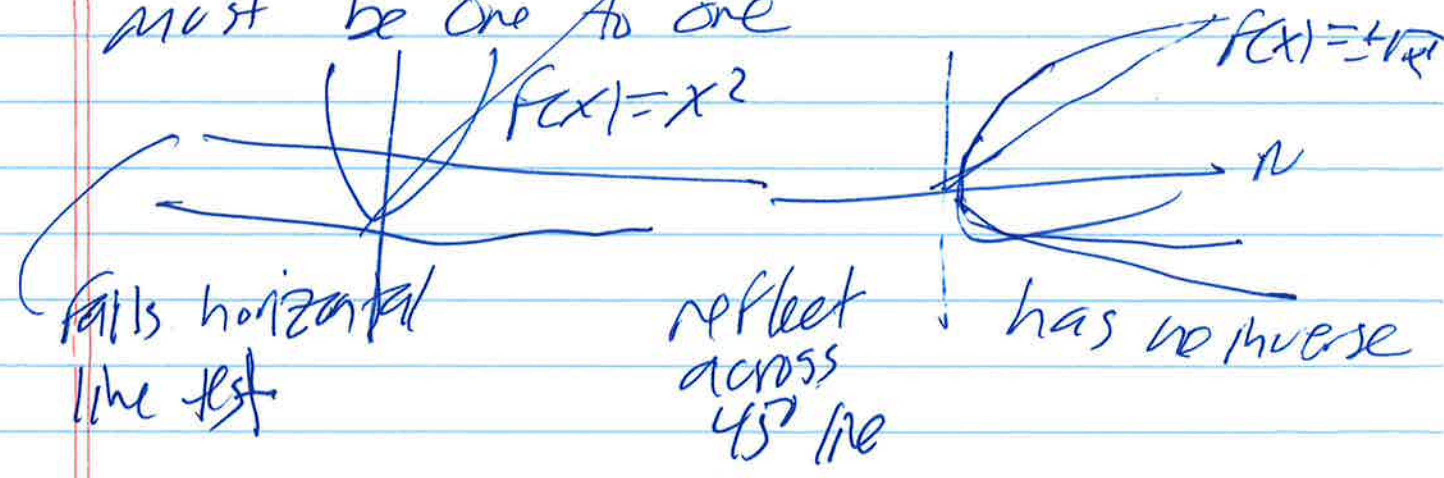
remember: "two students cant straddle one chair"

A function must obey vertical line test

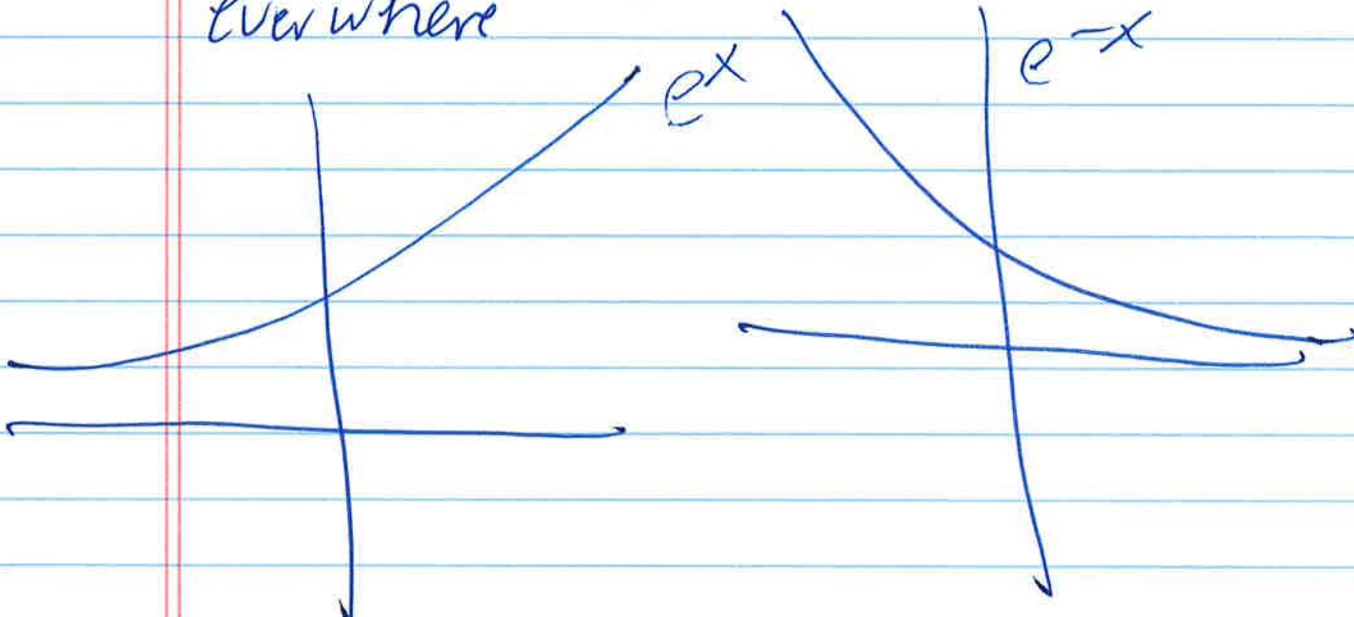
If it is true that each element of range ~~is gets only mapped to by only one element o~~ receives a mapping from only one element of range the function is one-to-one.

One-to-one functions satisfy both vertical line test and horizontal line test

For a function to have an inverse it must be one to one

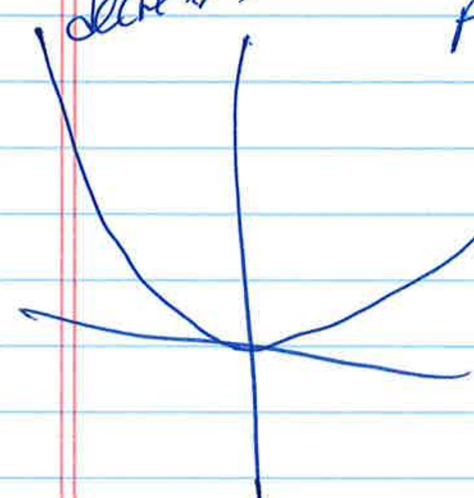


A real valued continuous function is one to one if and only if it is either increasing everwhere or decreasing everwhere



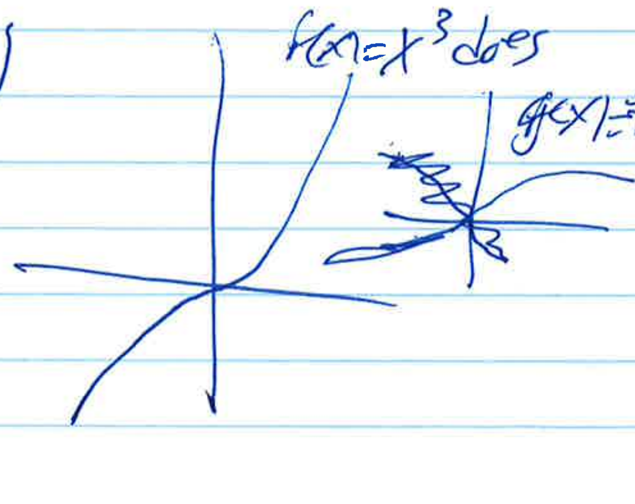
All exponentials with base $b>0$ $b\neq 1$

are 1 to 1. The inverse is $\log_b(x)$



$f(x)=x^2$ does not have an increase

$f(x)=x^3$ does



value of expression (1) approaches an irrational number that we call e. The irrational number e to 12 decimal places is

$$e = 2.718\,281\,828\,459$$

Compare this value of e with the value of e^1 from a calculator.

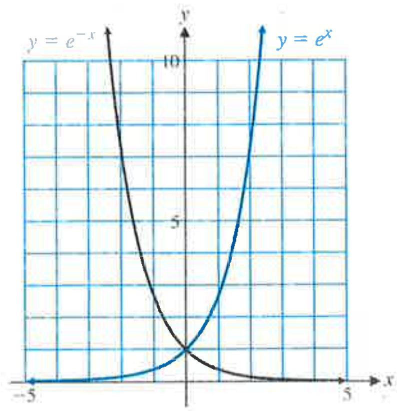
DEFINITION Exponential Function with Base e

Exponential function with base e and base $1/e$, respectively, are defined by

$$y = e^x \quad \text{and} \quad y = e^{-x}$$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$



Explore and Discuss 1 Graph the functions $f(x) = e^x$, $g(x) = 2^x$, and $h(x) = 3^x$ on the same set of coordinate axes. At which values of x do the graphs intersect? For positive values of x, which of the three graphs lies above the other two? Below the other two? How does your answer change for negative values of x? $x = 0$; if $x > 0$, $g(x) < f(x) < h(x)$; if $x < 0$, $h(x) < f(x) < g(x)$

Growth and Decay Applications

Functions of the form $y = ce^{kt}$, where c and k are constants and the independent variable t represents time, are often used to model population growth and radioactive decay. Note that if $t = 0$, then $y = c$. So the constant c represents the initial population (or initial amount). The constant k is called the **relative growth rate** and has the following interpretation: Suppose that $y = ce^{kt}$ models the population growth of a country, where y is the number of persons and t is time in years. If the relative growth rate is $k = 0.02$, then at any time t, the population is growing at a rate of $0.02y$ persons (that is, 2% of the population) per year.

We say that **population is growing continuously at relative growth rate k** to mean that the population y is given by the model $y = ce^{kt}$.

EXAMPLE 2 Exponential Growth Cholera, an intestinal disease, is caused by a cholera bacterium that multiplies exponentially. The number of bacteria grows continuously at relative growth rate 1.386, that is,

$$N = N_0 e^{1.386t}$$

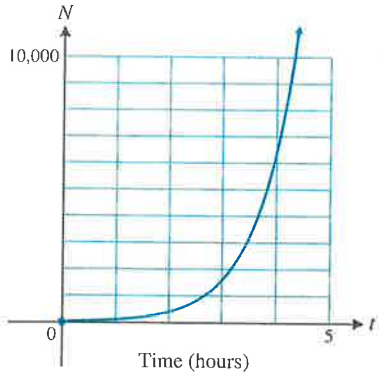


Figure 4

where N is the number of bacteria present after t hours and N_0 is the number of bacteria present at the start $(t = 0)$. If we start with 25 bacteria, how many bacteria (to the nearest unit) will be present:

(A) In 0.6 hour? (B) In 3.5 hours?

SOLUTION Substituting $N_0 = 25$ into the preceding equation, we obtain

$$N = 25e^{1.386t} \quad \text{The graph is shown in Figure 4.}$$

(A) Solve for N when $t = 0.6$:

$$\begin{aligned} N &= 25e^{1.386(0.6)} \quad \text{Use a calculator.} \\ &= 57 \text{ bacteria} \end{aligned}$$

(B) Solve for N when $t = 3.5$:

$$\begin{aligned} N &= 25e^{1.386(3.5)} \quad \text{Use a calculator.} \\ &= 3{,}197 \text{ bacteria} \end{aligned}$$

Matched Problem 2 Refer to the exponential growth model for cholera in Example 2. If we start with 55 bacteria, how many bacteria (to the nearest unit) will be present

(A) In 0.85 hour? (B) In 7.25 hours?

EXAMPLE 3 Exponential Decay Cosmic-ray bombardment of the atmosphere produces neutrons, which in turn react with nitrogen to produce radioactive carbon-14 (^{14}C). Radioactive ^{14}C enters all living tissues through carbon dioxide, which is first absorbed by plants. As long as a plant or animal is alive, ^{14}C is maintained in the living organism at a constant level. Once the organism dies, however, ^{14}C decays according to the equation

$$A = A_0e^{-0.000124t}$$

where A is the amount present after t years and A_0 is the amount present at time $t = 0$.

(A) If 500 milligrams of ^{14}C is present in a sample from a skull at the time of death, how many milligrams will be present in the sample in 15,000 years? Compute the answer to two decimal places.

Set MD Up

(B) The **half-life** of ^{14}C is the time t at which the amount present is one-half the amount at time $t = 0$. Use Figure 5 to estimate the half-life of ^{14}C.

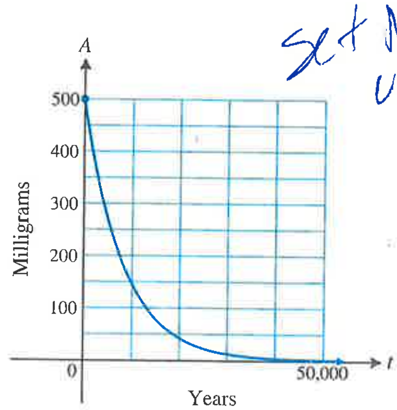


Figure 5

SOLUTION Substituting $A_0 = 500$ in the decay equation, we have

$$A = 500e^{-0.000124t} \quad \text{See the graph in Figure 5.}$$

(A) Solve for A when $t = 15{,}000$:

$$\begin{aligned} A &= 500e^{-0.000124(15{,}000)} \quad \text{Use a calculator.} \\ &= 77.84 \text{ milligrams} \end{aligned}$$

(B) Refer to Figure 5, and estimate the time t at which the amount A has fallen to 250 milligrams: $t \approx 6{,}000$ years. (Finding the intersection of $y_1 = 500e^{-0.000124x}$ and $y_2 = 250$ on a graphing calculator gives a better estimate: $t \approx 5{,}590$ years.)

Matched Problem 3 Refer to the exponential decay model in Example 3. How many milligrams of ^{14}C would have to be present at the beginning in order to have 25 milligrams present after 18,000 years? Compute the answer to the nearest milligram.

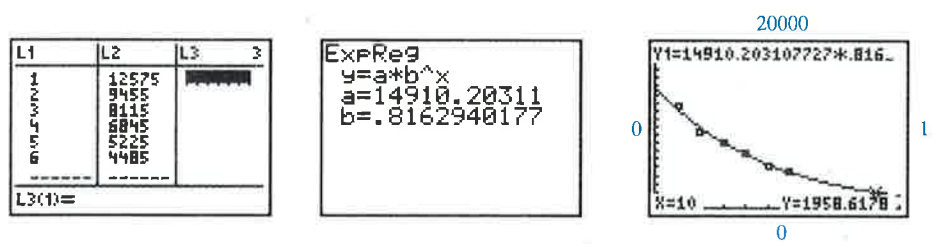
If you buy a new car, it is likely to depreciate in value by several thousand dollars during the first year you own it. You would expect the value of the car to decrease in each subsequent year, but not by as much as in the previous year. If you drive the car long enough, its resale value will get close to zero. An exponential decay function will often be a good model of depreciation; a linear or quadratic function would not be suitable (why?). We can use **exponential regression** on a graphing calculator to find the function of the form $y = ab^x$ that best fits a data set.

EXAMPLE 4 Depreciation Table 2 gives the market value of a hybrid sedan (in dollars) x years after its purchase. Find an exponential regression model of the form $y = ab^x$ for this data set. Estimate the purchase price of the hybrid. Estimate the value of the hybrid 10 years after its purchase. Round answers to the nearest dollar.

Table 2

x	Value ($)
1	12,575
2	9,455
3	8,115
4	6,845
5	5,225
6	4,485

SOLUTION Enter the data into a graphing calculator (Fig. 6A) and find the exponential regression equation (Fig. 6B). The estimated purchase price is $y_1(0) =$ \$14,910. The data set and the regression equation are graphed in Figure 6C. Using TRACE, we see that the estimated value after 10 years is \$1,959.



(A) (B) (C)

Figure 6

Matched Problem 4 Table 3 gives the market value of a midsize sedan (in dollars) x years after its purchase. Find an exponential regression model of the form $y = ab^x$ for this data set. Estimate the purchase price of the sedan. Estimate the value of the sedan 10 years after its purchase. Round answers to the nearest dollar.

Table 3

x	Value ($)
1	23,125
2	19,050
3	15,625
4	11,875
5	9,450
6	7,125

Compound Interest

The fee paid to use another's money is called **interest**. It is usually computed as a percent (called **interest rate**) of the principal over a given period of time. If, at the end of a payment period, the interest due is reinvested at the same rate, then the interest earned as well as the principal will earn interest during the next payment period. Interest paid on interest reinvested is called **compound interest** and may be calculated using the following compound interest formula:

If a **principal P (present value)** is invested at an annual **rate r** (expressed as a decimal) compounded m times a year, then the **amount A (future value)** in the account at the end of t years is given by

$$A = P\left(1 + \frac{r}{m}\right)^{mt} \quad \text{Compound interest formula}$$

For given r and m, the amount A is equal to the principal P multiplied by the exponential function b^t, where $b = (1 + r/m)^m$.

EXAMPLE 5 Compound Growth If $1,000 is invested in an account paying 10% compounded monthly, how much will be in the account at the end of 10 years? Compute the answer to the nearest cent.

SOLUTION We use the compound interest formula as follows:

$$\begin{aligned} A &= P\left(1 + \frac{r}{m}\right)^{mt} \\ &= 1{,}000\left(1 + \frac{0.10}{12}\right)^{(12)(10)} \quad \text{Use a calculator.} \\ &= \$2{,}707.04 \end{aligned}$$

The graph of

$$A = 1{,}000\left(1 + \frac{0.10}{12}\right)^{12t}$$

for $0 \le t \le 20$ is shown in Figure 7.

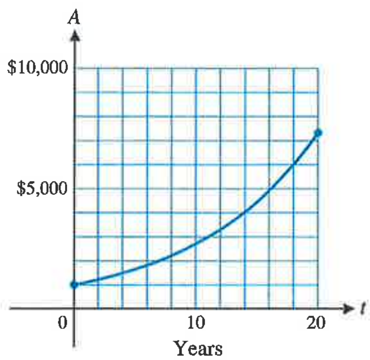


Figure 7

Matched Problem 5 If you deposit $5,000 in an account paying 9% compounded daily, how much will you have in the account in 5 years? Compute the answer to the nearest cent.

Explore and Discuss 2 Suppose that $1,000 is deposited in a savings account at an annual rate of 5%. Guess the amount in the account at the end of 1 year if interest is compounded (1) quarterly, (2) monthly, (3) daily, (4) hourly. Use the compound interest formula to compute the amounts at the end of 1 year to the nearest cent. Discuss the accuracy of your initial guesses. **1.** $1,050.95 **2.** $1,051.16 **3.** $1,051.27 **4.** $1,051.27

Explore and Discuss 2 suggests that if $1,000 were deposited in a savings account at an annual interest rate of 5%, then the amount at the end of 1 year would be less than $1,051.28, even if interest were compounded every minute or every second. The limiting value, approximately $1,051.271096, is said to be the amount in the account if interest were compounded continuously.

If a principal, P, is invested at an annual rate, r, and compounded continuously, then the amount in the account at the end of t years is given by

$$A = Pe^{rt} \quad \text{Continuous compound interest formula}$$

where the constant $e \approx 2.71828$ is the base of the exponential function.

EXAMPLE 6 Continuous Compound Interest If $1,000 is invested in an account paying 10% compounded continuously, how much will be in the account at the end of 10 years? Compute the answer to the nearest cent.

SOLUTION We use the continuous compound interest formula:

$$A = Pe^{rt} = 1000e^{0.10(10)} = 1000e = \$2{,}718.28$$

Compare with the answer to Example 5.

Matched Problem 6 If you deposit \$5,000 in an account paying 9% compounded continuously, how much will you have in the account in 5 years? Compute the answer to the nearest cent.

The formulas for compound interest and continuous compound interest are summarized below for convenient reference.

SUMMARY

Compound Interest: $A = P\left(1 + \dfrac{r}{m}\right)^{mt}$

Continuous Compound Interest: $A = Pe^{rt}$

where A = amount (future value) at the end of t years
P = principal (present value)
r = annual rate (expressed as a decimal)
m = number of compounding periods per year
t = time in years

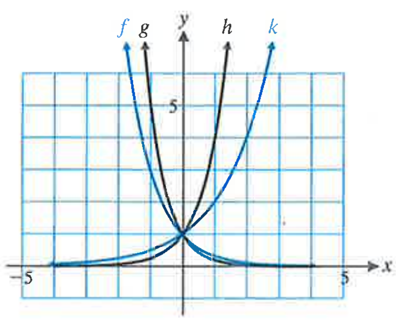
Exercises 2.5

A

1. Match each equation with the graph of f, g, h, or k in the figure.

(A) $y = 2^x$ k (B) $y = (0.2)^x$ g

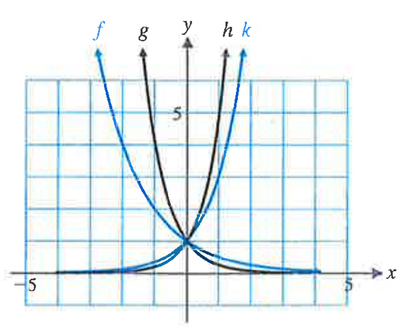
(C) $y = 4^x$ h (D) $y = \left(\frac{1}{3}\right)^x$ f



2. Match each equation with the graph of f, g, h, or k in the figure.

(A) $y = \left(\frac{1}{4}\right)^x$ g (B) $y = (0.5)^x$ f

(C) $y = 5^x$ h (D) $y = 3^x$ k



*Answer located in Additional Instructor's Answers section.

Graph each function in Problems 3–10 over the indicated interval.

3. $y = 5^x$; $[-2, 2]$ * **4.** $y = 3^x$; $[-3, 3]$ *

5. $y = \left(\frac{1}{5}\right)^x = 5^{-x}$; $[-2, 2]$ * **6.** $y = \left(\frac{1}{3}\right)^x = 3^{-x}$; $[-3, 3]$ *

7. $f(x) = -5^x$; $[-2, 2]$ * **8.** $g(x) = -3^{-x}$; $[-3, 3]$ *

9. $y = -e^{-x}$; $[-3, 3]$ * **10.** $y = -e^x$; $[-3, 3]$ *

B *In Problems 11–18, describe verbally the transformations that can be used to obtain the graph of g from the graph of f (see Section 2.2).*

11. $g(x) = -2^x$; $f(x) = 2^x$ *

12. $g(x) = 2^{x-2}$; $f(x) = 2^x$ *

13. $g(x) = 3^{x+1}$; $f(x) = 3^x$ *

14. $g(x) = -3^x$; $f(x) = 3^x$ *

15. $g(x) = e^x + 1$; $f(x) = e^x$ *

16. $g(x) = e^x - 2$; $f(x) = e^x$ *

17. $g(x) = 2e^{-(x+2)}$; $f(x) = e^{-x}$ *

18. $g(x) = 0.5e^{-(x-1)}$; $f(x) = e^{-x}$ *

19. Use the graph of f shown in the figure to sketch the graph of each of the following.

(A) $y = f(x) - 1$ * (B) $y = f(x + 2)$ *

(C) $y = 3f(x) - 2$ * (D) $y = 2 - f(x - 3)$ *

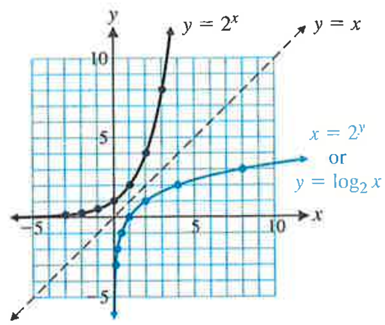


Figure 2

Exponential Function		Logarithmic Function	
x	$y = 2^x$	$x = 2^y$	y
-3	$\frac{1}{8}$	$\frac{1}{8}$	-3
-2	$\frac{1}{4}$	$\frac{1}{4}$	-2
-1	$\frac{1}{2}$	$\frac{1}{2}$	-1
0	1	1	0
1	2	2	1
2	4	4	2
3	8	8	3

Ordered pairs reversed

In general, since the graphs of all exponential functions of the form $f(x) = b^x$, $b \neq 1, b > 0$, are either increasing or decreasing, exponential functions have inverses.

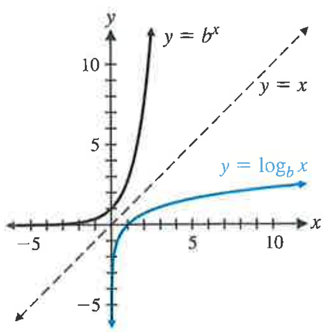


Figure 3

DEFINITION Logarithmic Functions

The inverse of an exponential function is called a **logarithmic function**. For $b > 0$ and $b \neq 1$,

Logarithmic form		Exponential form
$y = \log_b x$	is equivalent to	$x = b^y$

The **log to the base b of x** is the exponent to which b must be raised to obtain x. [*Remember:* A logarithm is an exponent.] The **domain** of the logarithmic function is the set of all positive real numbers, which is also the range of the corresponding exponential function; and the **range** of the logarithmic function is the set of all real numbers, which is also the domain of the corresponding exponential function. Typical graphs of an exponential function and its inverse, a logarithmic function, are shown in Figure 3.

CONCEPTUAL INSIGHT

Because the domain of a logarithmic function consists of the positive real numbers, the entire graph of a logarithmic function lies to the right of the y axis. In contrast, the graphs of polynomial and exponential functions intersect every vertical line, and the graphs of rational functions intersect all but a finite number of vertical lines.

The following examples involve converting logarithmic forms to equivalent exponential forms, and vice versa.

EXAMPLE 1 Logarithmic–Exponential Conversions Change each logarithmic form to an equivalent exponential form:

(A) $\log_5 25 = 2$ (B) $\log_9 3 = \frac{1}{2}$ (C) $\log_2\left(\frac{1}{4}\right) = -2$

SOLUTION

(A) $\log_5 25 = 2$	is equivalent to	$25 = 5^2$
(B) $\log_9 3 = \frac{1}{2}$	is equivalent to	$3 = 9^{1/2}$
(C) $\log_2\left(\frac{1}{4}\right) = -2$	is equivalent to	$\frac{1}{4} = 2^{-2}$

Matched Problem 1 Change each logarithmic form to an equivalent exponential form:

(A) $\log_3 9 = 2$ (B) $\log_4 2 = \frac{1}{2}$ (C) $\log_3\left(\frac{1}{9}\right) = -2$

EXAMPLE 2 Exponential–Logarithmic Conversions Change each exponential form to an equivalent logarithmic form:

(A) $64 = 4^3$ (B) $6 = \sqrt{36}$ (C) $\frac{1}{8} = 2^{-3}$

SOLUTION

(A) $64 = 4^3$ is equivalent to $\log_4 64 = 3$

(B) $6 = \sqrt{36}$ is equivalent to $\log_{36} 6 = \frac{1}{2}$

(C) $\frac{1}{8} = 2^{-3}$ is equivalent to $\log_2\left(\frac{1}{8}\right) = -3$

Matched Problem 2 Change each exponential form to an equivalent logarithmic form:

(A) $49 = 7^2$ (B) $3 = \sqrt{9}$ (C) $\frac{1}{3} = 3^{-1}$

To gain a deeper understanding of logarithmic functions and their relationship to exponential functions, we consider a few problems where we want to find x, b, or y in $y = \log_b x$, given the other two values. All values are chosen so that the problems can be solved exactly without a calculator.

EXAMPLE 3 Solutions of the Equation $y = \log_b x$ Find y, b, or x, as indicated.

(A) Find y: $y = \log_4 16$ (B) Find x: $\log_2 x = -3$

(C) Find b: $\log_b 100 = 2$

SOLUTION

(A) $y = \log_4 16$ is equivalent to $16 = 4^y$. So,

$$y = 2$$

(B) $\log_2 x = -3$ is equivalent to $x = 2^{-3}$. So,

$$x = \frac{1}{2^3} = \frac{1}{8}$$

(C) $\log_b 100 = 2$ is equivalent to $100 = b^2$. So,

$$b = 10$$ Recall that b cannot be negative.

Matched Problem 3 Find y, b, or x, as indicated.

(A) Find y: $y = \log_9 27$ (B) Find x: $\log_3 x = -1$

(C) Find b: $\log_b 1{,}000 = 3$

Properties of Logarithmic Functions

The properties of exponential functions (Section 2.5) lead to properties of logarithmic functions. For example, consider the exponential property $b^x b^y = b^{x+y}$. Let $M = b^x$, $N = b^y$. Then

$$\log_b MN = \log_b(b^x b^y) = \log_b b^{x+y} = x + y = \log_b M + \log_b N$$

So $\log_b MN = \log_b M + \log_b N$, that is, the logarithm of a product is the sum of the logarithms. Similarly, the logarithm of a quotient is the difference of the logarithms. These properties are among the eight useful properties of logarithms that are listed in Theorem 1.

THEOREM 1 Properties of Logarithmic Functions

If b, M, and N are positive real numbers, $b \neq 1$, and p and x are real numbers, then

1. $\log_b 1 = 0$

2. $\log_b b = 1$

3. $\log_b b^x = x$

4. $b^{\log_b x} = x, \quad x > 0$

5. $\log_b MN = \log_b M + \log_b N$

6. $\log_b \dfrac{M}{N} = \log_b M - \log_b N$

7. $\log_b M^p = p \log_b M$

8. $\log_b M = \log_b N$ if and only if $M = N$

EXAMPLE 4 Using Logarithmic Properties

(A) $\log_b \dfrac{wx}{yz} = \log_b wx - \log_b yz$
$= \log_b w + \log_b x - (\log_b y + \log_b z)$
$= \log_b w + \log_b x - \log_b y - \log_b z$

(B) $\log_b (wx)^{3/5} = \frac{3}{5}\log_b wx = \frac{3}{5}(\log_b w + \log_b x)$

(C) $e^{x \log_e b} = e^{\log_e b^x} = b^x$

(D) $\dfrac{\log_e x}{\log_e b} = \dfrac{\log_e (b^{\log_b x})}{\log_e b} = \dfrac{(\log_b x)(\log_e b)}{\log_e b} = \log_b x$

Matched Problem 4 Write in simpler forms, as in Example 4.

(A) $\log_b \dfrac{R}{ST}$ (B) $\log_b \left(\dfrac{R}{S}\right)^{2/3}$ (C) $2^{u \log_2 b}$ (D) $\dfrac{\log_2 x}{\log_2 b}$

The following examples and problems will give you additional practice in using basic logarithmic properties.

EXAMPLE 5 Solving Logarithmic Equations Find x so that

$$\tfrac{3}{2}\log_b 4 - \tfrac{2}{3}\log_b 8 + \log_b 2 = \log_b x$$

SOLUTION

$$\tfrac{3}{2}\log_b 4 - \tfrac{2}{3}\log_b 8 + \log_b 2 = \log_b x$$

$$\log_b 4^{3/2} - \log_b 8^{2/3} + \log_b 2 = \log_b x \quad \text{Property 7}$$

$$\log_b 8 - \log_b 4 + \log_b 2 = \log_b x$$

$$\log_b \frac{8 \cdot 2}{4} = \log_b x \quad \text{Properties 5 and 6}$$

$$\log_b 4 = \log_b x$$

$$x = 4 \quad \text{Property 8}$$

Matched Problem 5 Find x so that $3 \log_b 2 + \frac{1}{2}\log_b 25 - \log_b 20 = \log_b x$.

EXAMPLE 6 Solving Logarithmic Equations

Solve: $\log_{10} x + \log_{10}(x + 1) = \log_{10} 6$.

SOLUTION

$$\log_{10} x + \log_{10}(x + 1) = \log_{10} 6$$

$$\log_{10} [x(x + 1)] = \log_{10} 6 \quad \text{Property 5}$$

$$x(x + 1) = 6 \quad \text{Property 8}$$

$$x^2 + x - 6 = 0 \quad \text{Solve by factoring.}$$

$$(x + 3)(x - 2) = 0$$

$$x = -3, 2$$

We must exclude $x = -3$, since the domain of the function $\log_{10}(x + 1)$ is $x > -1$ or $(-1, \infty)$; so $x = 2$ is the only solution.

Matched Problem 6 Solve: $\log_3 x + \log_3(x - 3) = \log_3 10$.

Calculator Evaluation of Logarithms

Of all possible logarithmic bases, e and 10 are used almost exclusively. Before we can use logarithms in certain practical problems, we need to be able to approximate the logarithm of any positive number either to base 10 or to base e. And conversely, if we are given the logarithm of a number to base 10 or base e, we need to be able to approximate the number. Historically, tables were used for this purpose, but now calculators make computations faster and far more accurate.

Common logarithms are logarithms with base 10. **Natural logarithms** are logarithms with base e. Most calculators have a key labeled "log" (or "LOG") and a key labeled "ln" (or "LN"). The former represents a common (base 10) logarithm and the latter a natural (base e) logarithm. In fact, "log" and "ln" are both used extensively in mathematical literature, and whenever you see either used in this book without a base indicated, they will be interpreted as follows:

Common logarithm: $\log x$ means $\log_{10} x$
Natural logarithm: $\ln x$ means $\log_e x$

Finding the common or natural logarithm using a calculator is very easy. On some calculators, you simply enter a number from the domain of the function and press LOG or LN. On other calculators, you press either LOG or LN, enter a number from the domain, and then press ENTER. Check the user's manual for your calculator.

EXAMPLE 7 Calculator Evaluation of Logarithms Use a calculator to evaluate each to six decimal places:

(A) log 3,184 (B) ln 0.000 349 (C) $\log(-3.24)$

SOLUTION

(A) $\log 3{,}184 = 3.502\,973$

(B) $\ln 0.000\,349 = -7.960\,439$

(C) $\log(-3.24) =$ Error* -3.24 is not in the domain of the log function.

Matched Problem 7 Use a calculator to evaluate each to six decimal places:

(A) $\log 0.013\,529$ (B) $\ln 28.693\,28$ (C) $\ln(-0.438)$

Given the logarithm of a number, how do you find the number? We make direct use of the logarithmic-exponential relationships, which follow from the definition of logarithmic function given at the beginning of this section.

$$\log x = y \quad \text{is equivalent to} \quad x = 10^y$$
$$\ln x = y \quad \text{is equivalent to} \quad x = e^y$$

EXAMPLE 8 Solving $\log_b x = y$ for x Find x to four decimal places, given the indicated logarithm:

(A) $\log x = -2.315$ (B) $\ln x = 2.386$

SOLUTION

(A) $\log x = -2.315$ Change to equivalent exponential form.
$x = 10^{-2.315}$ Evaluate with a calculator.
$= 0.0048$

*Some calculators use a more advanced definition of logarithms involving complex numbers and will display an ordered pair of real numbers as the value of $\log(-3.24)$. You should interpret such a result as an indication that the number entered is not in the domain of the logarithm function as we have defined it.

(B) $\ln x = 2.386$ — Change to equivalent exponential form.

$x = e^{2.386}$ — Evaluate with a calculator.

$= 10.8699$

Matched Problem 8 Find x to four decimal places, given the indicated logarithm:

(A) $\ln x = -5.062$ (B) $\log x = 2.0821$

We can use logarithms to solve exponential equations.

EXAMPLE 9 **Solving Exponential Equations** Solve for x to four decimal places:

(A) $10^x = 2$ (B) $e^x = 3$ (C) $3^x = 4$

SOLUTION

(A)

$10^x = 2$	Take common logarithms of both sides.
$\log 10^x = \log 2$	Property 3
$x = \log 2$	Use a calculator.
$= 0.3010$	To four decimal places

(B)

$e^x = 3$	Take natural logarithms of both sides.
$\ln e^x = \ln 3$	Property 3
$x = \ln 3$	Use a calculator.
$= 1.0986$	To four decimal places

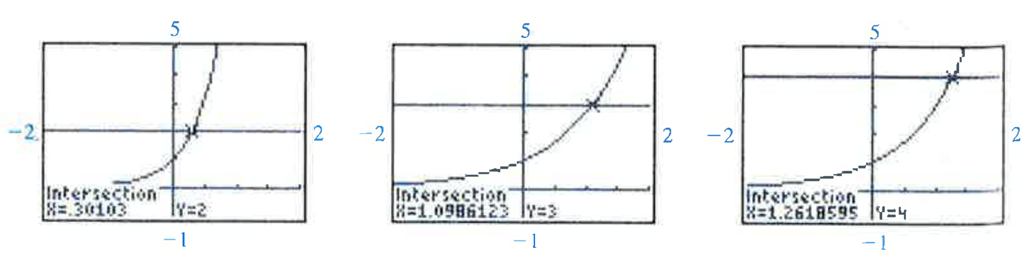
(C)

$3^x = 4$	Take either natural or common logarithms of both sides. (We choose common logarithms.)
$\log 3^x = \log 4$	Property 7
$x \log 3 = \log 4$	Solve for x.
$x = \dfrac{\log 4}{\log 3}$	Use a calculator.
$= 1.2619$	To four decimal places

Matched Problem 9 Solve for x to four decimal places:

(A) $10^x = 7$ (B) $e^x = 6$ (C) $4^x = 5$

Exponential equations can also be solved graphically by graphing both sides of an equation and finding the points of intersection. Figure 4 illustrates this approach for the equations in Example 9.



(A) $y_1 = 10^x$, $y_2 = 2$ (B) $y_1 = e^x$, $y_2 = 3$ (C) $y_1 = 3^x$, $y_2 = 4$

Figure 4 **Graphical solution of exponential equations**

Explore and Discuss 2 Discuss how you could find $y = \log_5 38.25$ using either natural or common logarithms on a calculator. [*Hint:* Start by rewriting the equation in exponential form.]
Find the intersection of $y_1 = 5^x$ and $y_2 = 38.25$.

Remark—In the usual notation for natural logarithms, the simplifications of Example 4, parts (C) and (D) on page 110, become

$$e^{x \ln b} = b^x \quad \text{and} \quad \frac{\ln x}{\ln b} = \log_b x$$

With these formulas, we can change an exponential function with base b, or a logarithmic function with base b, to expressions involving exponential or logarithmic functions, respectively, to the base e. Such **change-of-base formulas** are useful in calculus.

Applications

A convenient and easily understood way of comparing different investments is to use their **doubling times**—the length of time it takes the value of an investment to double. Logarithm properties, as you will see in Example 10, provide us with just the right tool for solving some doubling-time problems.

EXAMPLE 10 Doubling Time for an Investment How long (to the next whole year) will it take money to double if it is invested at 20% compounded annually?

SOLUTION We use the compound interest formula discussed in Section 2.5:

$$A = P\left(1 + \frac{r}{m}\right)^{mt} \qquad \text{Compound interest}$$

The problem is to find t, given $r = 0.20$, $m = 1$, and $A = 2p$; that is,

$$\begin{aligned}
2P &= P(1 + 0.2)^t \\
2 &= 1.2^t && \text{Solve for } t \text{ by taking the natural or} \\
1.2^t &= 2 && \text{common logarithm of both sides (we choose} \\
\ln 1.2^t &= \ln 2 && \text{the natural logarithm).} \\
t \ln 1.2 &= \ln 2 && \text{Property 7} \\
t &= \frac{\ln 2}{\ln 1.2} && \text{Use a calculator.} \\
&\approx 3.8 \text{ years} && [\textit{Note: } (\ln 2)/(\ln 1.2) \neq \ln 2 - \ln 1.2] \\
&\approx 4 \text{ years} && \text{To the next whole year}
\end{aligned}$$

When interest is paid at the end of 3 years, the money will not be doubled; when paid at the end of 4 years, the money will be slightly more than doubled.

Example 10 can also be solved graphically by graphing both sides of the equation $2 = 1.2^t$, and finding the intersection point (Fig. 5).

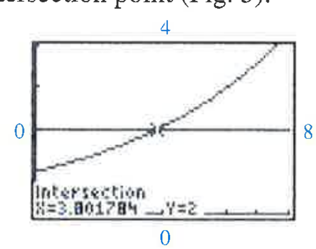


Figure 5 $y_1 = 1.2^x$, $y_2 = 2$

Matched Problem 10 How long (to the next whole year) will it take money to triple if it is invested at 13% compounded annually?