Math 151 - 2015 XB - W4 Day 3 Pg 1

Review properties of logarithms

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^x = x$,
4. $b^{\log_b x} = x$ must have $x > 0$
5. $\log_b MN = \log_b M + \log_b N$
6. $\log_b M/N = \log_b M - \log_b N$
7. $\log_b M^p = p \log_b M$
8. $\log_b M = \log_b N$ if and only if $M = N$

Review example

Write in simpler form (~~many~~ several logs of individual quantities)

$$\log_b \left(\frac{R}{S}\right)^{2/3}$$

$$= \frac{2}{3}\log_b(R) - \frac{2}{3}\log_b(S)$$

Example:

$e^{x \log_e b}$ Simplify

Want $e^{\log_e(\cdot)}$ or $\log_e(e^{(\cdot)})$
because these operations are inverses
(cancel each other out)

So move x up

$e^{\log_e b^x}$

Now cancel

$\boxed{b^x}$ answer

Try $2^{4 \log_2 b}$

$= b^4$

Eg 4D in §2.6

$$\frac{\log_e x}{\log_e b} = \frac{\log_e\left(b^{\log_b x}\right)}{\log_e b}$$

$$= \frac{\log_b x \log_e b}{\log_e b}$$

$$= \log_b x$$

$$\frac{\log_2 x}{\log_2 b} = \frac{\log_2 b^{\log_b(x)}}{\log_2 b}$$

$$= \frac{\log_b(x) \log_2 b}{\log_2 b}$$

$$= \log_b(x)$$

Example 5

Find x so that

$$\frac{3}{2}\log_b 4 - \frac{2}{3}\log_b 8 + \log_b 2 = \log_b x$$

$$\log_b 4^{3/2} - \log_b 8^{2/3} + \log_b 2 = \log_b x$$

$$\log_b 8 - \log_b 4 + \log_b 2 = \log_b x$$

$$\log_b\left(\frac{8}{4}\cdot 2\right) = \log_b(x)$$

$$\log_b 4 = \log_b x$$

$$x = 4$$

$\boxed{\text{Matched 5}}$

Find x so that

$$3\log_b 2 + \frac{1}{2}\log_b 25 - \log_b 20 = \log_b x$$

$$\log_b 2^3 + \log_b 25^{1/2} - \log_b 20 = \log_b x$$

$$\log_b\left(\frac{2^3 \cdot 25^{1/2}}{20}\right) = \log_b x$$

$$\log_b\left(\frac{8 \cdot 5}{20}\right) = \log_b x$$

$$\log_b\left(\frac{40}{20}\right) = \log_b x$$

$$\log_b 2 = \log_b x$$

$$x = 2$$

Example 6

Solve $\log_{10} x + \log_{10}(x+1) = \log_{10} 6$

$$\log_{10}(x)(x+1) = \log_{10} 6$$

$$(x)(x+1) = 6$$

$$x^2 + x - 6 = 0$$

$$(x+3)(x-2) = 0$$

$$x = -3, \; x = 2$$

But $\log_{10}(x+1)$ is not defined for $x = -3$

$\log_{10}(x+1)$ and $\log_{10} x$ are both defined for $x = 2$

Therefore the only solution is $x = 2$

Matched 6

Solve $\log_3 x + \log_3(x-3) = \log_3 10$

$$(x)(x-3) = 10$$
$$x^2 - 3x - 10 = 0$$
$$(x-5)(x+2) = 0$$

$x = -2$ or $x = +5$
Must be excluded

$$x = 5$$

Consider $\log_b x$, The bases that are most often used are

base 10 $\log_{10} x$ written $\log x$ without a base (10 implied)

base e $\log_e x$ written $\ln x$

base 10 called common logarithm

base e called natural logarithm.

log and ln on most calculators

Example 7

Use a calculator to evaluate

$\log 3{,}184$

$= 3.502973$

$\ln(0.000349)$

$= -7.960439$

$\log(-3{,}241)$

$=$ Error

8

A) Solve $\log x = -2.315$

$x = 10^{-2.315} = 0.0048$

B) Solve $\ln X = 2.386$

$X = e^{2.386} = 10.8699$

$\boxed{\text{Matched 8}}$

(A) Find x so that

$$\boxed{\ln x = -5.062}$$

$$x = e^{-5.062} = 0.006333$$

(B) Find x so that

$$\boxed{\log x = 2.0821}$$

$$x = 10^{2.0821} = 120.81$$

Eg 9

Solve (A) $10^x = 2$

$$x = \log 2 = 0.3010$$

(B) $e^x = 3$

$$x = \ln(3) = 1.0986$$

(C) $3^x = 4$

$$\log 3^x = \log 4$$

$$x = \frac{\log 4}{\log 3} = 1.2619$$

Matched 9

Solve for x

(A) $10^x = 7$

$x = \log 7 = 0.8471$

(B) $e^x = 6$

$x = \ln(6) = 1.791$

(C) $4^x = 5$

$x \log 4 = \log 5$

$x = \frac{\log 5}{\log 4} = 1.161$

If you invest your money, it can earn interest.

Interest can be paid in many different ways — eg. once per year or once per month.

Once you earn interest, the interest can earn interest — that is called compounding.

You may have noticed that banks offer accounts that differ in both interest rates and in compounding methods.

What is the difference between a bank offering 8% compounded ~~quarterly~~ annually (once per year) and a bank offering 8% compounded quarterly (4 times per year)?

Some offer annual compounding
Some quarterly
Some monthly
Some daily

$\boxed{\text{Annual compounding}}$

$$A = P(1+r)^t$$

Say you invest \$100 at 8% annual rate compounded annually

$P = 100$ ~~R~~ $= 0.08$

$$A = 100(1 + 0.08)^t$$

$$= 100(1.08) = \$108$$

After 1 year you have \$108

After 2 years you have

$$\underbrace{100(1.08)}_{\text{Amount after 1 year}}\underbrace{(1.08)}_{\text{compounded interest}} = 116.64$$

After 3 years you have

$$100(1.08)(1.08)(1.08) = 125.97$$

quarterly compounding

What happens when you invest \$100 at 8% interest compounded quarterly?

It is important to point out that what is meant by 8% is an annual interest rate that is applied quarterly (4 times) what this means is you divided 8% by 4 times to get 2% then compound 4x per year at this rate.

$100(1.02) = \$102$ after 1st quarter

$\uparrow$
P

$(100)((.02)(1.02) = \$104.04$ after 2nd quarter

$100(1.02)(1.02)(1.02) =$

106.12 after 3rd quarter

$\underbrace{100(1.02)(1.02)(1.02)(1.02) = 108.24}_{1\ \text{year}}$ after 4th quarter

So instead of having \$108.0 after 1 year you have \$108.24 under quarterly compounding

Every year you keep your money in the account your money grows by a factor of

$$(1.02)(1.02)(1.02)(1.02)$$

or

$$(1.02)^4$$

After t years your money grows by

$$A = P(1.02)^{4t}$$

Compounding m times per year you get

divide annual rate by m

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

compound m times per year for m years

important formula for this book

Compounding quarterly at a rate of 8%

$100 (1.02)(1.02)(1.02)(1.02)$ For one year

$=$

$100 (1.08243216)$

is the same as compounding annually for 1 year at a rate of 8.243216%

Compounding quarterly for t years at 8% is the same as compounding ~~an~~ annually for t-years at a rate of 8.243216%

8% is called the nominal annual interest rate

8.243216 is called the effective annual interest rate

No matter what ~~scheme~~ scheme bank use they will report

$$1 + r_{eff} = \left(1 + \frac{r}{m}\right)^m$$

The effective annual rate r_{eff} is related to the nominal annual rate r by this equation

Easy to see because then

$$A = \underbrace{P\left(1 + r_{eff}\right)^t}_{\text{annual equation compounded once}} = \underbrace{P\left(1 + \frac{r}{m}\right)^{mt}}_{\text{m times compounded annual equation}}$$

What is better compounded quarterly or compounded annually

It depends! ~~on the nominal rate~~
If the nominal rate remains the same then the better option is quarterly compounded
BUT Banks will often offer a lower nominal rate for quarterly compounding than for quarterly, Pick the one with the larger effective rate.

What happens when you crank up m (number of times compounding) and leave the nominal rate the same.

Remember we went from annual compounding to quarterly compounding and got

~~\$~~.29 extra.

How much can we get if we compound every microsecond? Can we get rich?

Turns out no

Amount of money you get will keep increasing but will increase less and less so that it will get closer and closer to (but never exceed a certain amount.

as $m \to \infty$ $$\left(1+\frac{r}{m}\right)^m \longrightarrow e^r$$

So that the maximum effective rate is

$$(1+r_{eff}) = e^r$$

$1 + r_{eff} = e^{0.08}$ $r_{eff} = e^{0.08} - 1 = 8.329$

\$8.32 is the most we will get from \$100 invested at annual rate 8% for 1 year

Here's a hint: Our book doesn't mention effective annual interest rates — at least not in chapter 2

So when you see an interest rate reported interpret it as a nominal interest rate for now.

I should say one other thing about effective rates.

The interest for quarterly and effective annual is equivalent when measured over the course of 1 year (or an integer number of years).

Quarterly compounding is not not the same over the course of the year

Even though you get the same amount of money at the end of the year, ~~you~~ (assuming the same effective annual rate)

For quarterly compounding you get that money earlier ~~than~~ in installments at the end of quarters rather than all at once at the end of the year.

Continuous compounding is the ~~[illegible]~~ limit as you crank ~~a~~ up m, you compound at every moment

$$A = Pe^{rt}$$

Where as compounding m times per year is

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Compounding annually

$$A = P(1+r)^{t}$$

Eg 5 ~~§2~~.5

If \$1000 is invested in an account paying 10% compounded monthly, how much will be in account at the end of 10 years

$$A = P\left(1+\frac{r}{m}\right)^{mt}$$

$P = 1000$
$r = .1$
$m = 12$
$t = 10$

$$= 1000\left(1+\frac{.1}{12}\right)^{12\cdot 10}$$

$$= 2707.04$$

Matched If you deposit \$5000 in an account paying 9% compounded daily how much will you have in account after 5 years

$$A = 5000\left(1+\frac{.09}{365}\right)^{365\cdot 5}$$

$$= \$7841.13$$

Eg 6 §2.5: If \$1000 is invested in an accant paying 10% compounded continuosly how much will be in account at the end of 10 years

$A = Pe^{rt}$

$P = 1000$
$r = 10\% = .1$
$t = 10$

$A = 1000 \cdot e^{(.1)(10)}$
$= 2718.28$

Matched 6 If you deposit \$5000 in an accant paying 9% compounded continuosly how much will you have in the account is 5 years

$A = Pe^{rt}$

$P = 5000$
$r = 9\% = 0.09$
$t = 5$

$\boxed{A = 5000e^{(0.09)(5)} = 7841.56}$

(just a litle more than answer to last matched problem same except continuous compounding instead of daily
(\$7841.13))